# Symmetric extensions of one-dimensional time changed minimal diffusions and multidimensional time changed RBMs

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Dirichlet forms and their geometry

Tohoku University, Sendai

1. Liouville property of one-dimensional minimal diffusions and their possible extensions
2. Reflections at infinity of time changed RBMs on a domain with (transient) Liouville branches
   - Beppo Levi space and extensions of time changed RBMs
   - Approaching probabilities of $Z$ and limits of BL-functions along $Z_t$
   - The maximal extension of a time changed RBM and the dimension of $\mathcal{H}^*(D)$
   - Partitions $\Pi$ of $F$ and all possible symmetric diffusion extensions $Y$ of a time changed RBM $X$

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(E; m)$ with an associated Hunt process $X = (X_t, \mathbf{P}_x)$ on $E$.

Let $\mathcal{F}_e$ and $\mathcal{F}^{\mathrm{ref}}$ be its extended Dirichlet space and its reflected Dirichlet space, repectively.

Then $\mathcal{F} \subset \mathcal{F}_e \subset \mathcal{F}^{\mathrm{ref}}$ and $\mathcal{E}$ is extended to both spaces.

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Then $\mathcal{F}\subset\mathcal{F}_e\subset\mathcal{F}^{\rm ref}$ and $\mathcal{E}$ is extended to both spaces.

Define the linear subspace $\mathcal{H}^*$ of $\mathcal{F}^{\rm ref}$ by

$$\mathcal{H}^*=\{u\in\mathcal{F}^{\rm ref}:\ \mathcal{E}(u,v)=0 \quad \text{for any} \quad v\in\mathcal{F}_e\}.$$

$\mathcal{H}^*$ is the collection of $X$-harmonic functions $u$ on $E$ of finite energy $\mathcal{E}(u,u)$.

We say that $\mathcal{E}$ enjoys the Liouville property if $\dim(\mathcal{H}^*)=1$

Yesterday we have considered

- Liouville Property of energy form on $\mathbb{R}^n$
- One-point reflection at infinity $\partial$ of a time changed diffusion $\check{X}$ associated with a strongly local transient Dirchlet form $\mathcal{E}$
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Today we shall consider

- Criteria for the Liouville property and transience of one-dimensional minimal diffusion
- All possible symmetric conservative diffusion extensions of time changed one-dimensional transient minimal diffusions
- All possible symmetric conservative diffusion extensions of time changed RBMs on a domain $D \subset \mathbb{R}^n$ with several Liouville branches ( a joint work with Z.-Q.Chen, to appear in Journal of Mathematical Socieity of Japan)

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We are revisiting the boundary problem of Markov processes initiated by Feller, Itô and McKean in 1950's.

# Liouville property of one-dimensional minimal diffusions and their possible extensions

$I = (r_1, r_2) \subset \mathbb{R}$

$s(x)$: continuous strictly increasing function on $I$ (canonical scale)

$$\mathcal{F}^{(s)} = \{u : \text{ abs. cont. on } I \text{ w.r.t. } ds \text{ and } \mathcal{E}^{(s)}(u,u) < \infty\}$$

where

$$\mathcal{E}^{(s)}(u,v) = \int_I \frac{du}{ds}(x)\frac{dv}{ds}(x)s(dx).$$

We call $r_i$ approachable if $|s(r_i)| < \infty$. $\quad i = 1, 2.$

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We call $r_i$ approachable if $|s(r_i)| < \infty$. $i = 1, 2$.

If $r_i$ is approachable, then
$u(r_i) = \lim_{x \in I, x \to r_i} u(x)$ exists and finite for any $u \in \mathcal{F}^{(s)}$. $i = 1, 2$.

Let

$$\mathcal{F}^{(s)}_{0,e} = \{u \in \mathcal{F}^{(s)} : \ u(r_i) = 0 \text{ if } r_i \text{ is approachable}\}.$$

For any positive Radon measure $m$ on $I$ with full support, we let

$$\mathcal{F}^{(s)}_{0} = \mathcal{F}^{(s)}_{0,e} \ \cap \ L^2(I; m).$$

Then $(\mathcal{E}^{(s)}, \mathcal{F}^{(s)}_0)$ is a strongly local irreducible regular Dirichlet form on $L^2(I; m)$. The associated diffusion on $I$ is called a minimal diffusion.

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**Theorem 1.1**

*$(\mathcal{E}^{(s)}, \mathcal{F}_0^{(s)})$ enjoys the Liouville property if and only if either $r_1$ or $r_2$ is non-approachable.*

**Proof.** Take any $u \in \mathcal{F}^{(s)} \ominus \mathcal{F}^{(s)}_{0,e}$:

$$u \in \mathcal{F}^{(s)}, \quad \mathcal{E}^{(s)}(u,v) = 0 \quad \text{for any } v \in \mathcal{F}^{(s)}_{0,e}.$$

**Proof.** Take any $u \in \mathcal{F}^{(s)} \ominus \mathcal{F}^{(s)}_{0,e}$:

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Write $w = \frac{du}{ds}$ and choose $v(x) = \psi(s(x)) \in \mathcal{F}^{(s)}_{0,e}$ for $\psi \in C_c^\infty(J),\ J = s(I)$. Then $\int_I w(x)\psi'(s(x))ds(x) = 0$ and

$$\int_J w(t(x))\psi'(x)dx = 0, \ \ t(x) = s^{-1}(x), \ \ \text{for any } \psi \in C_c^\infty(J),$$

which implies

$$w(t(x)) = C \text{ constant a.e. on } J, \quad w = \frac{du}{ds} = C \ ds\text{-a.e. on } I.$$

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$\mathcal{E}(u,u) = \int_I w(x)^2 ds(x) = C^2 s(I) < \infty.$

$s(I) = \infty \quad \Longrightarrow \quad C = 0, \quad u$ is constant

$s(I) < \infty \quad \Longrightarrow \quad u = C_1 + C_2\, s, \quad dim(\mathcal{H}^*) = 2.$

For the form $(\mathcal{E}^{(s)}, \mathcal{F}_0^{(s)})$:

(I) Liouville $\iff$ either $r_1$ or $r_2$ is non-approachable

(II) transient $\iff$ either $r_1$ or $r_2$ is approachable

(III) recurrent $\iff$ both $r_1$ and $r_2$ are non-approachable

(IV) Liouville and transient $\iff$

$r_1$ is approachable but $r_2$ is non-approachable

or, $r_2$ is approachable but $r_1$ is non-approachable

(V) non-Liouville and transient $\iff$

both $r_1$ and $r_2$ are approachable

(II) is proved in [CF2].

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Choose the canonical measure $m$ to be finite near approachable boundaries (regularization).

The minimal diffusion $X = (X_t, \zeta, \mathbf{P}_x)$ associated with $(\mathcal{E}^{(s)}, \mathcal{F}_0^{(s)})$ on $L^2(I; m)$ then approaches to approachable boundaries in finite time $\zeta$.

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We look for all possible symmetric conservative diffusion extensions $Y$ of $X$ beyond $\zeta$.

Case (IV) : $Y$ is unique.
$Y$ coincides with the one-point reflection $X^*$ of $X$ at the approachable boundary $r_1$ up to a homeomorphism.
$X^*$ may be viewed to live on the one-point compactification $I^*$ of $I$ (Theorem 5.1 of yesterday's talk)

Case (V) : Two possible extensions $Y$:
two point reflection at $r_1$ and $r_2$
one point extension to the one point compactification $I^*$ of $I$
(Theorem 4.1 of yesterday's talk)

# Beppo Levi space and extensions of time changed RBMs

For a domain $D \subset \mathbb{R}^d$, let us consider the spaces

$$\mathrm{BL}(D) = \{u \in L^2_{\mathrm{loc}}(D) : \ |\nabla u| \in L^2(D)\}, \quad H^1(D) = \mathrm{BL}(D) \cap L^2(D), \tag{2.1}$$

The space $\mathrm{BL}(D)$ called the Beppo Levi space was introduced by

J. Deny and J.L. Lions, Les espaces du type de Beppo Levi, *Ann. Inst. Fourier* **5**(1953/54), 305-370

as the space of Schwartz distributions whose first order derivatives are in $L^2(D)$, that can be identified with the function space described above.

The quotient space $\dot{\mathrm{BL}}(D)$ of $\mathrm{BL}(D)$ by the space of all constant functions on $D$ is a real Hilbert space with inner product

$$\mathbf{D}(u,v) = \int_D \nabla u(x) \cdot \nabla v(x) dx.$$

Define

$$(\mathcal{E}, \mathcal{F}) = (\frac{1}{2}\mathbf{D}, H^1(D)), \quad (\mathcal{F} \text{ is the domain of the form } \mathcal{E}), \qquad (2.2)$$

which is a Dirichlet form on $L^2(D)$.

The collection of those domains $D \subset \mathbb{R}^d$
for which (2.2) is regular on $L^2(\overline{D})$ will be denoted by $\mathcal{D}$.
$D \in \mathcal{D}$ if $D$ is either a domain of continuous boundary or
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an extendable domain relative to $H^1(D)$.

For $D \in \mathcal{D}$, the diffusion process $Z = (Z_t, \mathbf{Q}_x)$ on $\overline{D}$
associated with (2.2) is by definition
the reflecting Brownian motion (RBM)
which is known to be conservative always.

Further, the space $\mathrm{BL}(D)$ is nothing but
the reflected Dirichlet space of the form (2.2) ([CF2]).

The extended Dirichlet space of the Dirichlet form (2.2) is denoted by
$H^1_e(D)$
which is the completion of $H^1(D)$ relative to $\mathbf{D}$ in transient case.

Define

$$\mathcal{H}^*(D) = \{u \in \mathrm{BL}(D) : \ \mathbf{D}(u, v) = 0 \text{ for every } v \in H^1_e(D)\}. \tag{2.3}$$

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The Dirichlet form (2.2) or the RBM $Z$ is transient
only when $d \geq 3$ and $D$ is unbounded.

The transient RBM on $\overline{D}$ escapes to the point at infinity $\partial$ of $\overline{D}$;

$$\mathbf{Q}_x \left( \lim_{t \to \infty} Z_t = \partial \right) = 1 \quad \text{for q.e. } x \in \overline{D}.$$

To make the lifetime of the transient RBM $Z$ finite,
we take any strictly positive bounded function $f \in L^1(\overline{D}; m)$.
Then $A_t = \int_0^t f(Z_s)ds,\ t \geq 0$ is a strictly increasing PCAF of $Z$ with $\mathbf{E}_x^{\mathbf{Q}}[A_\infty] = Rf(x) < \infty$ for q.e. $x \in \overline{D}$.

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The time changed process $X = (X_t, \zeta, \mathbf{P}_x)$ of $Z$ by $A$ is defined by

$$X_t = Z_{\tau_t},\ t \geq 0, \quad \tau = A^{-1}, \quad \zeta = A_\infty, \quad \mathbf{P}_x = \mathbf{Q}_x,\ x \in \overline{D}.$$

Since $\mathbf{P}_x(\zeta < \infty,\ \lim_{t \to \zeta} X_t = \partial) = \mathbf{P}_x(\zeta < \infty) = 1 \quad \text{for q.e. } x \in \overline{D},$
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We denote $X$ also by $X^f$ to indicate its depedence on the function $f$.
For different choices of $f$, $X^f$ have a common geometric structure related each other only by time changes.

In this section, we consider the transient RBM $Z$ on the closure of a specfic unbounded domain $D \subset \mathbb{R}^d$ with $N$ number of Liouville branches.

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and to identify the family of all the corresponding extended Dirichlet spaces $(\mathcal{E}^Y, \mathcal{F}_e^Y)$ of $Y$'s

with the direct sums of the space $H_e^1(D)$ and subspaces of $\mathcal{H}^*(D)$ spanned by the approaching probabilities of $Z$ to the ends of Liouville branches in a simple manner.

This family is independent of the choice of $f$.

**Definition 2.1**

A domain $D \in \mathcal{D}$ is called a (transient) Liouville domain if the form (2.2) is transient and $\dim(\mathcal{H}^*(D)) = 1$.

A domain $D \in \mathcal{D}$ is a Liouville domain if and only if the form (2.2) is transient and any function $u \in \mathrm{BL}(D)$ admits a unique decomposition

$$u = u_0 + c, \qquad u_0 \in H^1_e(D), \quad c: \quad \text{a constant.} \tag{2.4}$$

The constant $c$ in the above will be denoted by $c(u)$.

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Another important example of a Liouville domain is provided by an unbounded uniform domain that has been shown by

P.W. Jones, Quasiconcormal mappings and extendability of functions in Sobolev spaces, *Acta Math.* **147**(1981),71-88

to be an extendable domain relative to the space $\mathrm{BL}(D)$.

A domain $D \subset \mathbb{R}^d$ is called a uniform domain

if there exists $C > 0$ such that for every $x, y \in D$, there is a rectifiable curve $\gamma$ in $D$ connecting $x$ and $y$ with length$(\gamma) \le C|x - y|$

$$\min\{|x-z|,\ |z-y|\} \le C\mathrm{dist}(z, D^c) \quad \text{for every } z \in \gamma.$$

It was proved in [CF1] that any unbounded uniform domain is a Liouville domain in the sense of Definition 2.1.

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It was proved in [CF1] that any unbounded uniform domain is a Liouville domain in the sense of Definition 2.1.

An unbounded uniform domain is such a domain that is fat in the middle and broaden toward the infinity.

The truncated infnite cone
$C_{A,a} = \{(r, \omega) : r > a,\ \omega \in A\} \subset \mathbb{R}^d$
for any connected open set $A \subset S^{d-1}$ with Lipschitz boundary is an unbounded uniform domain.

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To the contrary, (2.2) is recurrent for the cylinder
$D = \{(x, x') \in \mathbb{R}^d : x \in \mathbb{R},\ |x'| < 1\}$.

On the other hand, it has been shown in [CF2] that (2.2) is transient but $\dim(\mathcal{H}^*(D)) = 2$ for a special domain

$$D = B_1(\mathbf{0}) \cup \left\{(x, x') \in \mathbb{R}^d : x \in \mathbb{R},\ |x| > |x'|\right\}, \quad d \geq 3.$$

with two symmetric cone branches.

Here $B_r(\mathbf{0}),\ r > 0$, denotes an open ball with radius $r$ centered at the origin.

This domain is not uniform because of a presence of a bottleneck.

We shall consider much more general domains than this.

We shall work under the regularity condition

(A.1) $D$ is of a Lipschitz boundary $\partial D$,
in the sense formulated in

[FTo] M. Fukushima and M. Tomisaki, Construction and decomposition of reflecting diffusions on Lipschitz domains with Hölder cusps, *Probabb. Theory Relat. Fields* **106**(1996), 521-557

There exists then a conservative diffusion process $Z = (Z_t, \mathbf{Q}_x)$ on $\overline{D}$ associated with the regular Dirichlet form (2.2) on $L^2(\overline{D})$ whose resolvent $\{G_\alpha^Z; \alpha > 0\}$ has the strong Feller property in the sense that

$$G_\alpha^Z(bL^1(D)) \subset bC(\overline{D}).$$

$Z$ is a precise version of the RBM on $\overline{D}$.

In particular, the transition probability of $Z$ is absolutely continuous with respect to the Lebesgue measure.

Under the condition (A.1) and the transience assumption on (2.2), the RBM $Z = (Z_t, \mathbf{Q}_x)$ on $\overline{D}$ enjoys the properties that

$$\mathbf{Q}_x\left(\lim_{t\to\infty} Z_t = \partial\right) = 1 \quad \text{for every } x \in \overline{D}, \tag{2.5}$$

$$\mathbf{Q}_x\left(\lim_{t\to\infty} u(Z_t) = 0\right) = 1 \quad \text{for every } x \in \overline{D}, \tag{2.6}$$

for any $u \in H_e^1(D)$, $u$ being taken to be quasi-continuous ([CF2]).

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In the rest of my talk, we fix a domain $D$ of $\mathbb{R}^d$, $d \geq 3$, satisfying (A.1) and

(A.2) $$D \setminus \overline{B_r(\mathbf{0})} = \bigcup_{j=1}^{N} C_j$$

for some $r > 0$ and an integer $N$, where $C_1, \cdots, C_N$ are Liouville domains whose closures are mutually disjoint.

$D$ may be called a Lipschitz domain with $N$ number of Liouville branches.

Let $\partial_j$ be the point at infinity of the unbounded closed set $\overline{C}_j$ for each $1 \le j \le N$.

Let $F = \{\partial_1, \cdots, \partial_N\}$ and $\overline{D}^* = \overline{D} \cup F$.

$\overline{D}^*$ can be made to be a compact Hausdorff space if we employ as a local base of neighborhoods of each point $\partial_j \in F$ the neighborhoods of $\partial_j$ in $\overline{C}_j \cup \{\partial_j\}$.

Let $\partial_j$ be the point at infinity of the unbounded closed set $\overline{C}_j$ for each $1 \leq j \leq N$.

Let $F = \{\partial_1, \cdots, \partial_N\}$ and $\overline{D}^* = \overline{D} \cup F$.

$\overline{D}^*$ can be made to be a compact Hausdorff space if we employ as a local base of neighborhoods of each point $\partial_j \in F$ the neighborhoods of $\partial_j$ in $\overline{C}_j \cup \{\partial_j\}$.

$\overline{D}^*$ may be called the $N$-points compactification of $\overline{D}$.

Obviously the Dirichlet form (2.2) is transient for $D$.

# Approaching probabilities of $Z$ and limits of BL-functions along $Z_t$

For each $1 \le j \le N$, define the approaching probability of the RBM $Z = (Z_t, \mathbf{Q}_x)$ to $\partial_j$ by

$$\varphi_j(x) = \mathbf{Q}_x \left( \lim_{t \to \infty} Z_t = \partial_j \right), \quad x \in \overline{D}. \tag{2.7}$$

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**Proposition 2.2**

*It holds that*

$$\sum_{j=1}^{N} \varphi_j(x) = 1 \quad \textit{for every} \quad x \in \overline{D}, \tag{2.8}$$

*and, for each* $1 \leq j \leq N$,

$$\varphi_j(x) > 0 \quad \textit{for every} \quad x \in \overline{D}. \tag{2.9}$$

(2.8) is an immediate consequence of (2.5).

## Proposition 2.3

*For* $u \in \mathrm{BL}(D)$*, let* $c_j(u) = c(u\big|_{C_j})$*,* $1 \le j \le N$*. Then*

$$\mathbf{Q}_x \left( Z_{\infty-} = \partial_j,\ \lim_{t\to\infty} u(Z_t) = c_j(u) \right) = \mathbf{Q}_x \left( Z_{\infty-} = \partial_j \right),\ x \in \overline{D}. \tag{2.10}$$

*If* $c_j(u) = 0$ *for every* $1 \le j \le N$*, then* $u \in H^1_e(D)$*.*

This essentially follows from (2.6) combined with (2.4).

# The maximal extension of a time changed RBM and the dimension of $\mathcal{H}^*(D)$

Fix a strictly positive bounded integrable function $f$ on $\overline{D}$ and define $A_t = \int_0^t f(Z_s)ds, \quad t \geq 0.$

$A_t$ is a positive continuous additive functional (PCAF) of the RBM $Z = (Z_t, \mathbf{Q}_x)$ on $\overline{D}$ in the strict sense with full support. Let

$$X_t = Z_{\tau_t}, \quad \tau = A^{-1}, \quad \zeta = A_\infty, \quad \mathbf{P}_x = \mathbf{Q}_x \text{ for } x \in \overline{D}.$$

$X = (X_t, \zeta, \mathbf{P}_x)$ is a diffusion process on $\overline{D}$ satisfying

$$\mathbf{P}_x(\zeta < \infty, \ \lim_{t \uparrow \zeta} X_t = \partial) = 1, \quad \forall x \in \overline{D}.$$

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$X = (X_t, \zeta, \mathbf{P}_x)$ is a diffusion process on $\overline{D}$ satisfying

$$\mathbf{P}_x(\zeta < \infty, \ \lim_{t \uparrow \zeta} X_t = \partial) = 1, \quad \forall x \in \overline{D}.$$

We rewrite the approaching probability $\varphi_j$ of $Z$ to $\partial_j$ as

$$\varphi_j(x) = \mathbf{P}_x\left(\zeta < \infty, \quad X_{\zeta-} = \partial_j\right), \quad x \in \overline{D}, \quad 1 \leq j \leq N. \qquad (2.11)$$

We consider the problem of extending $X$ after $\zeta$, particularly, from $\overline{D}$ to its $N$-points compactification $\overline{D}^* = \overline{D} \cup F$ for $F = \{\partial_1, \cdots, \partial_N\}$.

$X$ is symmetric with respect to the measure $m(dx) = f(x)dx$ and its Dirichlet form $(\mathcal{E}^X, \mathcal{F}^X)$ on $L^2(\overline{D}; m)$ is given by

$$\mathcal{E}^X = \frac{1}{2}\mathbf{D}, \qquad \mathcal{F}^X = H_e^1(D) \cap L^2(\overline{D}; m). \tag{2.12}$$

Since the extended Dirichlet space and the reflected Dirichlet space are invariant under a time change by a fully supported PCAF ([CF2]).

these spaces for $\mathcal{E}^X$ are still given by $H_e^1(D)$ and $\mathrm{BL}(D)$, respectively.

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$m(dx) = f(x)dx$ is extended from $\overline{D}$ to $\overline{D}^*$ by setting $m(F) = 0$.

An $m$-symmetric conservative diffusion process $X^*$ on $\overline{D}^*$ will be called a symmetric conservative diffusion extension of $X$
if its part process on $\overline{D}$ is equivalent in law with $X$.

## Proposition 2.4

*There exists a unique symmetric conservative diffusion extension* $X^*$ *of* $X$ *from* $\overline{D}$ *to* $\overline{D}^* = \overline{D} \cup F$.

$X^*$ *is recurrent.*

*Let* $(\mathcal{E}^*, \mathcal{F}^*)$ *and* $\mathcal{F}_e^*$ *be the Dirichlet form of* $X^*$ *on* $L^2(\overline{D}^*, m)$ $(= L^2(D; m))$ *and its extended Dirichlet space, respectively. Then*

$$\mathcal{F}_e^* = H_e^1(D) \oplus \left\{ \sum_{j=1}^{N} c_j \varphi_j \ : \ c_j \in \mathbb{R} \right\} \subset \mathrm{BL}(D), \tag{2.13}$$

$$\mathcal{E}^*(u, v) = \frac{1}{2} \mathbf{D}(u, v), \qquad u, v \in \mathcal{F}_e^*. \tag{2.14}$$

This proposition can be proved using Proposition 2.2 and applying a general existence theorem of a many-point extension formulated in [CF2]
to the $m$-symmetric diffusion $X$ on $\overline{D}$ and
to the $N$-points compactification $\overline{D}^* = \overline{D} \cup F$ of $\overline{D}$.

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to the $N$-points compactification $\overline{D}^* = \overline{D} \cup F$ of $\overline{D}$.

This proposition particularly implies that
$\{\varphi_i : 1 \leq i \leq N\} \subset \mathcal{H}^*(D)$.

Proposition 2.3 associates constants $c_i(u),\ 1 \leq i \leq N$, for each $u \in \mathrm{BL}(D)$,

Considering the bounded martingale $\{\varphi_i(Z_t)\}$, it can be also verified that

$$c_j(\varphi_i) = \delta_{ij}, \qquad 1 \leq i, j \leq N.$$

which means that $\{\varphi_i, 1 \leq i \leq N\}$ are linearly independent and

any $u \in \mathcal{H}^*(D)$ can be written as $u = \sum_{i=1}^N c_i(u)\varphi_i$.

**Theorem 2.5**

$\dim(\mathcal{H}^*(D)) = N$ *and*

$$\mathcal{H}^*(D) = \left\{ \sum_{j=1}^{N} c_j \, \varphi_j \ : \ c_j \in \mathbb{R} \right\}. \tag{2.15}$$

*The $m$-symmetric conservative diffusion extension $X^*$ of the time changed RBM $X$ constructed in* Proposition 2.4 *is the maximal extension of $X$ in the sense that the extended Dirichlet space $(\mathcal{F}_e^*, \mathcal{E}^*)$ of $X^*$ equals $(\mathrm{BL}(D), \frac{1}{2}\mathbf{D})$ the reflected Dirichlet space of $X$.*

# Partitions $\Pi$ of $F$ and all possible symmetric diffusion extensions $Y$ of a time changed RBM $X$

We continue to consider the $N$-points compactification
$\overline{D}^* = \overline{D} \cup F,\ F = \{\partial_1, \cdots, \partial_N\},$ of $\overline{D}$
A map $\Pi$ from $F$ onto a finite set $\widehat{F} = \{\widehat{\partial}_1, \cdots, \widehat{\partial}_\ell\}$ with $\ell \le N$
is called a partition of $F$. We let $\overline{D}^{\Pi,*} = \overline{D} \cup \widehat{F}$.
Extend the map $\Pi$ from $F$ to $\overline{D}^*$ by setting $\Pi x = x,\ x \in \overline{D}$, and
introduce the quotient topology on $\overline{D}^{\Pi,*}$ by $\Pi$,
in other words, we employ as the family of open subsets of $\overline{D}^{\Pi,*}$
$\mathcal{U}_\Pi = \{U \subset \overline{D}^{\Pi,*} : \Pi^{-1}(U)$ is an open subset of $\overline{D}^*\}$

$\overline{D}^{\Pi,*}$ is a compact Hausdorff space and may be called

an $\ell$-points compactification of $\overline{D}$ obtained from $\overline{D}^*$
by identifying the points in the set $\Pi^{-1}\widehat{\partial}_i \subset F$
as a single point $\widehat{\partial}_i$ for each $1 \le i \le \ell$.

Given a partition $\Pi$ of $F$, the approaching probabilities $\widehat{\varphi}_i$ of the RBM $Z = (Z_t, \mathbf{Q}_x)$ to $\widehat{\partial}_i \in \widehat{F}$ are defined by

$$\widehat{\varphi}_i(x) = \sum_{j \in \Pi^{-1}\widehat{\partial}_i} \varphi_j(x), \quad x \in \overline{D}, \quad 1 \leq i \leq \ell. \tag{2.16}$$

As before, we consider the time changed process $X = (X_t, \zeta, \mathbf{P}_x)$ on $\overline{D}$ of $Z$ by a strictly positive bounded integrable function $f$ on $\overline{D}$.

$m(dx) = f(x)dx$ is extended from $\overline{D}$ to $\overline{D}^{\Pi,*}$ by setting $m(\widehat{F}) = 0$.

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$m(dx) = f(x)dx$ is extended from $\overline{D}$ to $\overline{D}^{\Pi,*}$ by setting $m(\widehat{F}) = 0$.

Just as in Proposition 2.4, there exists then a unique $m$-symmetric conservative diffusion extension $X^{\Pi,*}$ of $X$ from $\overline{D}$ to $\overline{D}^{\Pi,*}$.

The Dirichlet form $(\mathcal{E}^{\Pi,*}, \mathcal{F}^{\Pi,*})$ of $X^{\Pi,*}$ on $L^2(\overline{D}^{\Pi,*}; m)$ $(= L^2(D; m))$ admits the extended Dirichlet space $(\mathcal{F}_e^{\Pi,*}, \mathcal{E}^{\Pi,*})$ expressed as

$$\mathcal{F}_e^{\Pi,*} = H_e^1(D) \oplus \left\{ \sum_{i=1}^{\ell} c_i \widehat{\varphi}_i \ : \ c_i \in \mathbb{R} \right\} \subset \mathrm{BL}(D), \tag{2.17}$$

$$\mathcal{E}^{\Pi,*}(u,v) = \frac{1}{2}\,\mathbf{D}(u,v), \qquad u, v \in \mathcal{F}_e^{\Pi,*}. \tag{2.18}$$

We now claim that

the family $\{\overline{X}^{\Pi,*} : \Pi$ is a partition of $F\}$ exhausts all possible $m$-symmetric conservative diffusion extensions of the time changed RBM $X$ on $\overline{D}$.

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Let $E$ be a Lusin space into which $\overline{D}$ is homeomorpically embedded as an open subset.

The measure $m(dx) = f(x)dx$ on $\overline{D}$ is extended to $E$ by setting $m(E \setminus \overline{D}) = 0$.

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The measure $m(dx) = f(x)dx$ on $\overline{D}$ is extended to $E$ by setting $m(E \setminus \overline{D}) = 0.$

Let $Y = (Y_t, \mathbf{P}_x^Y)$ be an $m$-symmetric conservative diffusion process on $E$ whose part process on $\overline{D}$ is identical in law with $X$.

We denote by $(\mathcal{E}^Y, \mathcal{F}^Y)$ and $\mathcal{F}_e^Y$ the Dirichlet form of $Y$ on $L^2(E; m)$ and its extended Dirichlet space.

We call $Y$ an $m$-symmetric conservative diffusion extension of $X$.

**Theorem 2.6**

*There exists a partition $\Pi$ of $F$ such that, as Dirichlet forms on $L^2(\overline{D}; m)$,*

$$(\mathcal{E}^Y, \mathcal{F}^Y) = (\mathcal{E}^{\Pi,*}, \mathcal{F}^{\Pi,*}). \tag{2.19}$$

*$Y$ is a quasi-homeomorphic image of $X^{\Pi,*}$.*

Z.-M. Ma and M. Röckner, *Introduction to the Theory of (non-symmetric) Dirichlet forms*, Springer, 1992

Z.-Q. Chen, Z.-M. Ma and M. Röckner, Quasi-homeomorphisms of Dirichlet forms, *Nagoya Math. J* **136**(1994), 1-15

enable us to assume that

$(\mathcal{E}^Y, \mathcal{F}^Y)$ is a regular Dirichlet form on $L^2(E; m)$,

$Y$ is an associated Hunt process on $E$ and

$\widetilde{F} = E \setminus \overline{D}$ is quasi-closed subset of $E$.

The fact that $Y$ is a conservative diffusion extension of $X$ then implies

$$\begin{cases} H_e^1(D) \subset \mathcal{F}_e^Y \subset \mathrm{BL}(D), \quad \mathcal{H}^Y := \{\mathbf{H}u : u \in \mathcal{F}_e^Y\} \subset \mathcal{H}^*(D), \\ \mathcal{E}^Y(u,u) = \frac{1}{2}\mathbf{D}(u,u) + \frac{1}{2}\mu^c_{\langle \mathbf{H}u \rangle}(\widetilde{F}), \quad u \in \mathcal{F}_e^Y, \end{cases} \tag{2.20}$$

where $\mathbf{H}u(x) = \mathbf{E}_x^Y[u(Y_{\sigma_{\widetilde{F}}})],\ x \in E.$

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where $\mathbf{H}u(x) = \mathbf{E}_x^Y[u(Y_{\sigma_{\widetilde{F}}})],\ x \in E$.

Actually it can be verified that

$$\mu^c_{\langle u \rangle}(\widetilde{F}) = 0 \quad \text{for any } u \in \mathcal{H}^Y. \tag{2.21}$$

Indeed, by (2.20) and Theorem 2.5, any $u \in \mathcal{H}^Y$ has an expression $\sum_{j=1}^N c_j \varphi_j$
and so $u(Y_{\sigma_{\widetilde{F}}})$ can take only finite number of values $c_j, 1 \le j \le N$,
yielding that $u(\xi)$ can take only those values q.e. on $\widetilde{F}$.
Hence we have (2.21) due to the energy measure density theorem of Bouleau-Hirsch.

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Hence we have (2.21) due to the energy measure density theorem of Bouleau-Hirsch.

We let $\partial_i \sim \partial_j$ if $u(\partial_i) = u(\partial_j)$ for all $u \in \mathcal{H}^Y$.
This gives the partition $\Pi$ of $F$, yielding the identification (2.19).

Since the both hand sides of (2.19) are quasi-regular Dirichlet spaces, they are related by a quasi-homeomorphism of the underlying spaces.

## Remark 2.7

(Symmetric diffusion for a uniformly elliptic differential operator)
Given measurable functions $a_{ij}(x)$, $1 \le i, j \le d$, on $D$ such that
$a_{ij}(x) = a_{ji}(x), \quad \Lambda^{-1}|\xi|^2 \le \sum_{1\le i,j\le d} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \quad x \in D,\ \xi \in \mathbb{R}^d$,
for some constant $\Lambda \ge 1$, we consider a Dirichlet form

$$(\mathcal{E}, \mathcal{F}) = (\mathbf{a}, H^1(D)) \tag{2.22}$$

on $L^2(D)$ where
$\mathbf{a}(u, v) = \int_D \sum_{i,j=1}^d a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) dx, \quad u, v \in H^1(D).$

If we replace the Dirichlet form (2.2) on $L^2(D)$ and the assoicated RBM $Z$ on $\overline{D}$, respectively,
by the Dirichlet form (2.22) on $L^2(D)$ and the associated reflecting diffusion process on $\overline{D}$ constructed in [FTo]

all results in this talk remain valid without any essential change.

We have studied all possible conservative symmetric diffusion extensions of the RBM on a domain of several (transient) Liouville branches.

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This setting bears a strong similarity to
connected sums of non-parabolic manifolds studied in

Y. Kuz'menko and S. Molchanov, Counterexamples to Liouville-type theorems, *Moscow Univ. Math. Bull.* **34**(1979), 35-39

A. Grigor'yan and L. Salloff-Coste, Heat kernels on manifolds with ends, *Ann. Inst. Fourier, Grenoble* **59**(2009), 1917-1997

although the main concern in these papers was the heat kernel estimates.