Liouville property of harmonic functions of finite energy for Dirichlet forms

Masatoshi Fukushima (Osaka)

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Dirichlet forms and their geometry

Tohoku University, Sendai

The title of my tomorrow's talk will be changed

from

Reflections at infinity of time changed RBMs on a domain with Liouville branches

to

Symmetric extensions of
one-dimensional time changed minimal diffusions
and multidimensional time changed RBMs

Some generalities

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(E; m)$ with an associated Hunt process $X = (X_t, \mathbf{P}_x)$ on E. Let $\mathcal{F}_e$ and $\mathcal{F}^{\mathrm{ref}}$ be its extended Dirichlet space and its reflected Dirichlet space, repectively.

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For $f \in \mathcal{F}_{\rm loc}$, define using the Beurling-Deny decomposition

$$\widetilde{\mathcal{E}}(f,f) = \frac{1}{2}\mu^c_{\langle f\rangle}(E) + \frac{1}{2}\int_{E\times E}(f(x)-f(y))^2 J(dx,dy) + \int_E f(x)^2\kappa(dx) \le \infty$$

Let $(\tau_k u)(x) = ((-k) \vee u(x)) \wedge k,\ x \in E.$ Then $(\mathcal{F}^{\rm ref}, \mathcal{E})$ is defined as

$$\begin{cases} \mathcal{F}^{\rm ref} = \left\{ u :\ |u| < \infty[m],\ \tau_k u \in \mathcal{F}_{\rm loc},\ \forall k \geq 1,\ \sup_{k\geq 1} \widetilde{\mathcal{E}}(\tau_k u, \tau_k u) < \infty \right\} \\ \mathcal{E}(u,u) = \lim_{k\to\infty} \widetilde{\mathcal{E}}(\tau_k u, \tau_k u) \quad \text{for} \quad u \in \mathcal{F}^{\rm ref}. \end{cases}$$

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which is extended to any quasi-regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ in the recent book[CF2].

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which is extended to any quasi-regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ in the recent book[CF2].

$\mathcal{F}^{\rm ref} = \mathcal{F}_e$ if $(\mathcal{E}, \mathcal{F})$ is recurrent.

Define the linear subspace $\mathcal{H}^*$ of $\mathcal{F}^{\rm ref}$ by

$$\mathcal{H}^* = \{u \in \mathcal{F}^{\rm ref} : \ \mathcal{E}(u,v) = 0 \quad \text{for any} \quad v \in \mathcal{F}_e\}.$$

$\mathcal{H}^*$ is the collection of X-harmonic functions u on E of finite energy $\mathcal{E}(u,u)$.
We will be concerned with a specific Liouville property

$$\dim(\mathcal{H}^*) = 1 \tag{1.1}$$

of the form $\mathcal{E}$ and its probabilistic significance.

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We first give two general remarks on the Liouville property (1.1).
A Borel function h on E is said to be X-harmonic
if it is specified and finite up to quasi equivalence and
if for every relatively compact open subset $G \subset E$, $\mathbf{E}_x[|h(X_{\tau_G})|] < \infty$
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and $h(x) = \mathbf{E}_x[h(X_{\tau_G}]$ for q.e. $x \in E$,
where τ_G denotes the first exit time from G. By the next proposition, we only need to consider the transient form $\mathcal{E}$ to study the Liouville property (1.1).

Proposition 1.1

(i) *If $\mathcal{E}$ is irreducible and recurrent, then $\mathcal{E}$ enjoys the property* (1.1).

(ii) *If $\mathcal{E}$ is transient and if any bounded X-harmonic function on E is constant, then $\mathcal{E}$ enjoys the property* (1.1).

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For an Euclidean domain $D \subset \mathbb{R}^n$,

the Beppo Levi space and the Sobolev space of order $(1, 2)$ are defined, respectively, by

$$\mathrm{BL}(D) = \{u \in L^2_{\mathrm{loc}}(D) : \ |\nabla u| \in L^2(D)\}, \quad H^1(D) = \mathrm{BL}(D) \cap L^2(D). \tag{1.2}$$

Let $\mathbf{D}(u, v) = \int_D \nabla u(x) \cdot \nabla v(x) dx, \quad u, v \in \mathrm{BL}(D)$.

The space $\mathrm{BL}(D)$ is just the space of Schwartz distributions whose first order derivatives are in $L^2(D)$.

It was introduced and profoundly studied by Deny-Lions [DL. 1953] following the preceding works by Beppo Levi [L, 1906], Nikodym [N, 1933] and Deny [De1, 1950].

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Later on, the space BL(D) was designated as $L_2^1(D)$ by Maz'ja [M,1985] and studied in a more general context of the spaces $L_p^\ell(D)$ for $p > 0$ and integers ℓ.

However the space $\mathrm{BL}(D)$ bears its own independent potential theoretic and probabilistic significances from the beginning.

See [De1], Brelot[Br,1953], Doob[Do,1962], Fukushima[F1,1969] and Deny[De2,1970] in this connection.

Now suppose a domain $D \subset \mathbb{R}^n$ is either of continuous boundary or an extendable domain relative to $H^1(D)$.
The symmetric form $\mathcal{E}$ with $\mathcal{D}(\mathcal{E}) = \mathcal{F}$ defined by

$$\mathcal{E} = \frac{1}{2}\mathbf{D}, \qquad \mathcal{F} = H^1(D), \tag{1.3}$$

is then a regular strongly local irreducible Dirichlet form on $L^2(\overline{D})$ and the associated diffusion X on $\overline{D}$ is by definition the reflecting Brownian motion (RBM in abbreviation).

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The extended Dirichlet space of $\mathcal{E}$ is denoted by $H_e^1(D)$ and called the extended Sobolev space of order 1.

$\mathrm{BL}(D)$ is nothing but the reflected Dirichlet space of this form $\mathcal{E}$ ([CF2]).

The space $\mathcal{H}^* = \mathrm{BL}(D) \ominus H_e^1(D)$ consists of those functions on D with finite Dirichlet integral such that they are not only harmonic on D in the ordinary sense but also their quasi continuous versions are harmonic with respect to the RBM Z on $\overline{D}$.

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On the other hand, it can be demonstrated that $\dim(\mathcal{H}^*) = N$ when $n \geq 3$ and D is a Lipschitz domain with N number of Liouville branches in the sense formulated in tomorrow's talk.

In the simplest case that $D = \mathbb{R}^n$ the whole space, $\mathcal{H}^*$ is just the space of harmonic functions on $\mathbb{R}^n$ with finite Dirichlet integrals.

Brelot [Br,1953] first observed that the property (1.1) is valid, namely, any harmonic function on $\mathbb{R}^n$ with finite Dirichlet integral is constant.

A simple question arises:

(Q) Is the property (1.1) still valid for the whole space $\mathbb{R}^n$ and for more general Dirichlet forms than $\frac{1}{2}\mathbf{D}$?

Liouville property of energy form on $\mathbb{R}^n$

Consider a measurable function $\rho(x)$ on $\mathbb{R}^n$ such that

$$0 < \lambda_\ell \le \rho(x) \le \Lambda_\ell < \infty, \quad \text{for every } x \in B_\ell := \{|x| < \ell\}, \quad \ell > 0. \tag{2.1}$$

for constants $\lambda_\ell,\ \Lambda_\ell$ depending on $\ell > 0$,

and the associated spaces $\mathcal{F}^\rho$, $\mathcal{G}^\rho$ and form $\mathbf{D}^\rho$ defined respectively by

$$\mathcal{F}^\rho = \{u \in L^2(\mathbb{R}^n; \rho dx)\ :\ |\nabla u| \in L^2(\mathbb{R}^n; \rho dx)\}, \tag{2.2}$$

$$\mathcal{G}^\rho = \{u \in L^2_{\rm loc}(\mathbb{R}^n)\ :\ |\nabla u| \in L^2(\mathbb{R}^n; \rho dx)\}, \tag{2.3}$$

$$\mathbf{D}^\rho(u,v) = \int_{\mathbb{R}^n} \nabla u(x) \cdot \nabla v(x)\ \rho(x)\ dx. \tag{2.4}$$

Proposition 2.1

(i) *The energy form* $\mathcal{E}^\rho = (\mathbf{D}^\rho, \mathcal{F}^\rho)$ *is a regular strongly local and irreducible Dirichlet form on* $L^2(\mathbb{R}^n; \rho dx)$.

(ii) *The quotient space* $\dot{\mathcal{G}}^\rho$ *of the weighted Beppo Levi space* $\mathcal{G}^\rho$ *by constant functions is a Hilbert space with inner product* $\mathbf{D}^\rho$.

(iii) *Let* $(\mathcal{F}_e^\rho, \mathcal{E}^\rho)$ *be the extended Dirichlet space of the energy form* $(\mathcal{E}^\rho \mathcal{F}^\rho)$. *Then* $\quad \mathcal{F}_e^\rho \subset \mathcal{G}^\rho$ *and* $\mathcal{E}^\rho(u,u) = \mathbf{D}^\rho(u,u), \quad u \in \mathcal{F}_e^\rho$.

(iv) *Let* $(\mathcal{F}^{\rho,\mathrm{ref}}, \mathcal{E}^{\rho,\mathrm{ref}})$ *be the reflected Dirichlet space of the energy form* $\mathcal{E}^\rho$. *Then* $\quad \mathcal{F}^{\rho,\mathrm{ref}} = \mathcal{G}^\rho, \quad \mathcal{E}^{\rho,\mathrm{ref}} = \mathbf{D}^\rho$.

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Corollary 2.2

If $\lambda_\ell, \Lambda_\ell$ *are independent of* $\ell > 1.$, *then the energy form* $\mathcal{E}^\rho$ *has the Loiuville property* (2.1).

because $\mathcal{F}_e^\rho = H_e^1(\mathbb{R}^n)$, $\mathcal{G}^\rho = \mathrm{BL}(\mathbb{R}^n)$ in this case.

Theorem 2.3

Let $\rho(x) = \eta(|x|),\ x \in \mathbb{R}^n$,
for a positive C^∞*-function* η *on* $[0, \infty)$
such that η *is constant on* $[0, \epsilon)$ *for some* $\epsilon > 0$.
Then the energy form $\mathcal{E}^\rho$ *satisfies the Liouville property* (1.1) *when* $n \geq 2$.
When $n = 1$, *dim*$(\mathcal{H}^*) = 2$ *in transient case.*

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Proof. In view of Proposition 1.1 and Proposition 2.1, it suffices to consider only the transient case in order to verify the Liouville property (1.1).
According to the first edition of [FOT], $\mathcal{E}^\rho$ is transient if and only if

$$\int_1^\infty \frac{1}{\eta(r)r^{n-1}} dr < \infty. \tag{2.5}$$

In what follows, we assume that η satisfies condition (2.5).

We use the polar coordinate

$$x_1 = r\cos\theta_1,\ x_2 = r\sin\theta_1\cos\theta_2,\ x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3, \cdots ,$$

$$x_{n-1} = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1},$$

$$x_n = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}.$$

Then, for $u, v \in C_c^1(\mathbb{R}^n)$,

$$\mathbf{D}^\rho(u,v) = \tag{2.6}$$
$$\iint \left[u_r v_r + \frac{u_{\theta_1}v_{\theta_1}}{r^2} + \frac{u_{\theta_2}v_{\theta_2}}{r^2\sin^2\theta_1} + \cdots + \frac{u_{\theta_{n-1}}v_{\theta_{n-1}}}{r^2\sin^2\theta_1\cdots\sin^2\theta_{n-2}}\right]$$
$$\times\eta(r)r^{n-1}\sin^{n-2}\theta_1\cdots\sin\theta_{n-2}drd\theta_1\cdots d\theta_{n-1}.$$

For a C^∞-function u on $\mathbb{R}^n$, we denote by $I_\eta(u,u)$ the value of the integral of the right hand side of (2.6) for $v = u$.

By noting that $\rho(x) = \eta(|x|)$ is a C^∞-function on $\mathbb{R}^n$, we let

$$Lu(x) = \Delta u(x) + \nabla \log \rho(x) \cdot \nabla u(x), \quad x \in \mathbb{R}^n. \tag{2.7}$$

We say that u is $\mathcal{E}^\rho$-harmonic if

$$u \in C^\infty(\mathbb{R}^n), \quad Lu(x) = 0, \ x \in \mathbb{R}^n.$$

u belongs to the space $\mathcal{H}^* = \mathcal{G}^\rho \ominus \mathcal{F}_e^\rho$ if and only if

$$u \text{ is } \mathcal{E}^\rho\text{-harmonic and} \quad I_\eta(u,u) < \infty. \tag{2.8}$$

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Now take any function $u \in \mathcal{H}^*$. We can then derive from the transience condition (2.5) that

$$u_{\theta_k} = 0, \qquad 1 \le k \le n-1. \tag{2.9}$$

Thus $u \in \mathcal{H}^*$ depends only on r and, in terms of a scale function $ds(r) = \dfrac{dr}{\eta(r) r^{n-1}}$ on $(0, \infty)$,
the conditions (2.7) on u is reduced to

$$I_\eta(u,u) = \sigma_n \int_0^\infty \left(\frac{du(r)}{ds(r)} \right)^2 ds(r) < \infty, \quad Lu(r) = \frac{1}{r^{n-1}} \frac{d}{dr} \cdot \frac{du(r)}{ds(r)} = 0,$$

so that $\frac{du(r)}{ds(r)}$ equals a constant C
and $I_\eta(u,u) = \sigma_n C^2 s(0,\infty) < \infty$.

When $n \geq 2$, $s(0,\infty) = \infty$ and we get $C = 0$, yielding that u is constant. $\square$

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We conjecture that the energy form $\mathcal{E}^\rho$ on $\mathbb{R}^n$ always satisfies the Liouville property when $n \geq 2$ under the local uniform ellipticity (2.1).

Strongly local transient Dirichlet form $\mathcal{E}$ and a time change $\check{X}$ of the associated diffusion

From now on, we fix a general transient and strongly local Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$.
$\mathcal{F}_e$, $\mathcal{F}^{\rm ref}$ denote its extended and reflected Dirichlet space, respectively.

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From now on, we fix a general transient and strongly local Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$.
$\mathcal{F}_e$, $\mathcal{F}^{\rm ref}$ denote its extended and reflected Dirichlet space, respectively.
Let $X = (X_t, \zeta, \mathbf{P}_x)$ be the associated diffusion process on E.
The lifetime ζ of X can be finite or infinite.
Since X admits no killing inside E,

$$\mathbf{P}_x(\lim_{t\to\zeta} X_t = \partial) = 1 \quad \text{q.e.} \quad x \in E, \tag{3.1}$$

$$\mathbf{P}_x(\lim_{t\to\zeta} u(X_t) = 0) = 1 \quad \text{q.e.} \quad x \in E, \tag{3.2}$$

where ∂ is the point at infinity of E and u is any quasi continuous function belonging to the extended Dirichlet space $\mathcal{F}_e$.

Fix an arbitrary positive finite measure ν on E charging no $\mathcal{E}$-polar set such the the quasi-support of ν equals E.

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A typical example of such a measure ν is $\nu(dx) = f(x)m(dx)$ for a strictly positive Borel function f on E with $\int_E f dm < \infty$ and, in this case, $A_t = \int_0^{t\wedge\zeta} f(X_s)ds,\ t \geq 0$.

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Let $\check{X} = (\check{X}_t, \check{\zeta}, \mathbf{P}_x)$ be the time changed process of X by means of A:

$$\check{X}_t = X_{\tau_t}, \quad \tau_t = \inf\{s : A_s > t\}, \qquad \check{\zeta} = A_\zeta.$$

$\check{X}$ is a diffusion process on E symmetric with respect to the measure ν and the Dirichlet form $\check{\mathcal{E}} = (\check{\mathcal{E}}, \check{\mathcal{F}})$ of $\check{X}$ on $L^2(E;\nu)$ is given by

$$\check{\mathcal{E}} = \mathcal{E}, \qquad \check{\mathcal{F}} = \mathcal{F}_e \cap L^2(E;\nu), \tag{3.3}$$

which is strongly local and regular.

Proposition 3.1

(i) *It holds that*

$$\mathbf{P}_x(\check{\zeta} < \infty,\ \lim_{t \uparrow \check{\zeta}} \check{X}_t = \partial) = \mathbf{P}_x(\check{\zeta} < \infty) = 1 \text{ for q.e. } x \in E. \tag{3.4}$$

(ii) *The extended and reflected Dirichlet spaces of* $(\check{\mathcal{E}}, \check{\mathcal{F}})$ *equal* $(\mathcal{F}_e, \mathcal{E})$ *and* $(\mathcal{F}^{\mathrm{ref}}, \mathcal{E}^{\mathrm{ref}})$, *respectively.*

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Since the lifetime $\check{\zeta}$ of the time changed diffusion $\check{X}$ is finite $\mathbf{P}_x$-a.s. for q.e. $x \in E$ by the above lemma,

the boundary problem concerning possible Markovian extensions of $\check{X}$ beyond its lifetime $\check{\zeta}$ makes a perfect sense.

Proposition 3.1

(i) *It holds that*

$$\mathbf{P}_x(\check{\zeta} < \infty, \lim_{t \uparrow \check{\zeta}} \check{X}_t = \partial) = \mathbf{P}_x(\check{\zeta} < \infty) = 1 \text{ for q.e. } x \in E. \tag{3.4}$$

(ii) *The extended and reflected Dirichlet spaces of* $(\check{\mathcal{E}}, \check{\mathcal{F}})$ *equal* $(\mathcal{F}_e, \mathcal{E})$ *and* $(\mathcal{F}^{\rm ref}, \mathcal{E}^{\rm ref})$, *respectively.*

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the boundary problem concerning possible Markovian extensions of $\check{X}$ beyond its lifetime $\check{\zeta}$ makes a perfect sense.

For different choices of ν, the diffusions $\check{X}$ share a common geometric structure related each other only by time changes.

So the study of the boundary problem for $\check{X}$ as we shall engage in the rest of my talks is a good way to make a closer look at the behavior of the diffusion process X around ∂.

One-point reflection of $\check{X}$ at ∂

Denote by E^* the one-point compactification $E \cup \{\partial\}$ of E.

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Denote by E^* the one-point compactification $E \cup \{\partial\}$ of E.

The measure ν is extended from E to E^* by setting $\nu(\{\partial\}) = 0$.

We construct a ν-symmetric conservative diffusion extension of $\check{X}$ from E to E^* by constructing a regular strongly local Dirichlet form on $L^2(E^*; \nu)$.
Note that $L^2(E^*; \nu)$ can be identified with $L^2(E; \nu)$.

Due to the strong locality of $\mathcal{E}$ and the definition of its reflected Dirichlet space $\mathcal{F}^{\rm ref}$, we have

$$1 \in \mathcal{F}^{\rm ref}, \qquad \mathcal{E}^{\rm ref}(1,1) = 0. \tag{4.1}$$

Furthermore $\mathcal{F}_e$ does not contain a non-zero constant function because of the transience of $\mathcal{E}$.

In what follows, every function in the space $\mathcal{F}_e$ is taken to be $\mathcal{E}$-quasi-continuous.
Let us define

$$\begin{cases} \mathcal{F}_e^* = \{u + c : u \in \mathcal{F}_e,\ c \in \mathbb{R}\}, \\ \mathcal{E}^*(u_1 + c_1, u_2 + c_2) = \mathcal{E}(u_1, u_2),\ u_i \in \mathcal{F}_e,\ c_i \in \mathbb{R},\ i = 1, 2. \end{cases} \tag{4.2}$$

$(\mathcal{F}_e^*, \mathcal{E}^*)$ is a subspace of $(\mathcal{F}^{\mathrm{ref}}, \mathcal{E}^{\mathrm{ref}})$.

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$(\mathcal{F}_e^*, \mathcal{E}^*)$ is a subspace of $(\mathcal{F}^{\mathrm{ref}}, \mathcal{E}^{\mathrm{ref}})$.

Theorem 4.1

(i) *Define* $\check{\mathcal{F}}^* = \mathcal{F}_e^* \cap L^2(E; \nu)$.
The form $\check{\mathcal{E}}^* = (\mathcal{E}^*, \check{\mathcal{F}}^*)$ *is then a regular strongly local Dirichlet form on* $L^2(E^*; \nu)$.
(ii) *The extended Dirichlet space of* $\check{\mathcal{E}}^*$ *equals* $(\mathcal{F}_e^*, \mathcal{E}^*)$. $\check{\mathcal{E}}^*$ *is recurrent.*
(iii) *Let* $\check{X}^* = (\check{X}_t^*, \mathbf{P}_x^*)$ *be the diffusion process on* E^* *associated with* $\check{\mathcal{E}}^*$.
The part of $\check{X}^*$ *on* E *being killed upon hitting* ∂ *is then identical in law with the time changed diffusion* $\check{X}$.
$\check{X}^*$ *is conservative and irreducible.*

We call $\check{X}^*$ the one-point reflection of $\check{X}$ at ∂.

The first construction of such a one-point reflection at ∂ goes back to [F1, 1969] where $\check{X} = X$ and X was the absorbing Brownian motion on an arbitrary bounded domain of $\mathbb{R}^n$.

This theorem generalizes a theorem in Fukushima-Tanaka[FT, 2005] where a Poincaré inequality for $\mathcal{E}$ was assumed.

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There is an alternative quite different way to construct a one-point reflection $\check{X}^*$ of $\check{X}$ at ∂
by using a Poisson point process of excursions of $\check{X}$ around ∂,

which makes the structure of the constructed process $\check{X}^*$
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by using a Poisson point process of excursions of $\check{X}$ around ∂,

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but requires a certain regularity condition on the resolvent of $\check{X}$
in the construction ([FT], [CF2]).

Notice that $\check{X}^*$ becomes irreducible, while $\check{X}$ may not be.

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$\check{X}^*$ connects at ∂ the irreducible components E_i of $\check{X}$

by the following entrance law $\mu_t(B)$ to piece together

the excursions of $\check{X}$ on each E_i around ∂:

$$\int_0^t \mu_t(B)ds = \int_B \check{\mathbf{P}}_x(\check{\zeta} \leq t)\nu(dx), \quad B \subset E = \bigcup_i E_i.$$

Liouville property of $\mathcal{E}$ and uniqueness of a symmetric conservative diffusion extension of $\check{X}$

Let $\widehat{E}$ be a Lusin space into which E is homeomorpically embedded as an open subset.
The measure ν on E is extended to $\widehat{E}$ by setting $\nu(\widehat{E} \setminus E) = 0$.
Let $Y = (Y_t, \zeta^Y, \mathbf{P}_x^Y)$ be any ν-symmetric conservative diffusion process on $\widehat{E}$ whose part process on E being killed upon leaving E is identical in law with $\check{X}$.
We denote by $(\mathcal{E}^Y, \mathcal{F}^Y)$ the Dirichlet form of Y on $L^2(\widehat{E}; \nu)$.
We call Y a ν-symmetric consevative diffusion extension of $\check{X}$.

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Theorem 5.1

Suppose $\mathcal{E}$ satisfies the Liouville property (1.1). *Then*
(i) *As Dirichlet forms on $L^2(E, \nu)$,*

$$(\mathcal{E}^Y, \mathcal{F}^Y) = (\mathcal{E}^*, \check{\mathcal{F}}^*). \tag{5.1}$$

(ii) *A quasi-homeomorphic image of Y is identical with $\check{X}^*$.*

Proof. $\mathcal{E}^Y$ is a quasi-regular Dirichlet form on $L^2(\widehat{E};\nu)$ and Y is properly associated with it.

By the transfer method, we can therefore assume that $\widehat{E}$ is a locally compact separable metric space,
ν is a fully supported positive Radon measure on $\widehat{E}$,
$(\mathcal{E}^Y, \mathcal{F}^Y)$ is a regular Dirichlet form on $L^2(\widehat{E};\nu)$ and Y is an associated diffusion Hunt process on $\widehat{E}$.
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Since $\check{X}$ is the part on E of Y, we can characterize its Dirichlet form (3.3) as

$$\check{\mathcal{F}} = \{u \in \mathcal{F}^Y : u = 0 \quad \text{q.e. on} \quad \widehat{E} \setminus E\}, \quad \check{\mathcal{E}} = \mathcal{E}^Y \text{ on } \check{\mathcal{F}} \times \check{\mathcal{F}}.$$

This means that $\check{\mathcal{F}}$ is an ideal of $\mathcal{F}^Y$;
if $u \in \check{\mathcal{F}}_b,\ v \in \mathcal{F}_b^Y$, then $uv \in \check{\mathcal{F}}_b$,
in other words, $\mathcal{F}^Y$ is a Silverstein extension of $\check{\mathcal{F}}$.

We can then invoke Theorem 6.6.9 in [CF2] about

the maximality of the reflected Dirichlet space among Silverstein extensions

under the condition that the original Dirichlet form admits no killing inside, namely, $\kappa = 0$ in its Beurling-Deny decomposition.

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This condition is missing in [CF2].

The reflected Dirichlet space of $\check{\mathcal{E}}$ equals $(\mathcal{F}^{\rm ref}, \mathcal{E}^{\rm ref})$ by virtue of Proposition 3.1 (ii). Thus

$$\mathcal{F}^Y \ \subset \ \mathcal{F}_a^{\rm ref} \ (= \mathcal{F}^{\rm ref} \cap L^2(E;\nu)),$$

But under the present asumption of the Liouville property (1.1).

$$\mathcal{F}^{\rm ref} = \mathcal{F}_e^*, \quad \mathcal{E}^{\rm ref} = \mathcal{E}^*, \quad \mathcal{F}_a^{\rm ref} = \check{\mathcal{F}}^*, \tag{5.2}$$

so that $\mathcal{F}^Y \subset \check{\mathcal{F}}^*$.

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$$\mathcal{F}^{\rm ref} = \mathcal{F}_e^*, \quad \mathcal{E}^{\rm ref} = \mathcal{E}^*, \quad \mathcal{F}_a^{\rm ref} = \check{\mathcal{F}}^*, \tag{5.2}$$

so that $\mathcal{F}^Y \subset \check{\mathcal{F}}^*$. As Y is assumed to be conservative while $\check{X}$ has a finite liftime by Proposition 3.1 (i), $\check{\mathcal{F}}$ is a proper subspace of $\mathcal{F}^Y$.
Hence we must have the identity $\mathcal{F}^Y = \check{\mathcal{F}}^*$.
Since Y is a diffusion with no killing inside $\widehat{E}$, the regular Dirichlet form $(\mathcal{E}^Y, \mathcal{F}^Y)$ is strongly local so that $\mathcal{E}^Y(1,1) = 0$, yielding
$\mathcal{E}^Y(w,w) = \mathcal{E}^Y(u,u) = \mathcal{E}(u,u) = \mathcal{E}^*(w,w)$, for $w = u + c$, $u \in \check{\mathcal{F}}$,