Excursion measure away from an exit boundary of one-dimensional diffusion processes

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Watanabe [2] has discovered the necessary and sufficient condition that the ratio of the occupation time on the positive side of a one-dimensional generalized diffusion process converges in law to some non-trivial random variable. In the positively recurrent cases, in particular, the limit random variable is a constant.

Recently Kasahara and Watanabe [1] have studied the scaling limit of the fluctuation in the positively recurrent cases. In their context, they obtained the following convergence theorem: The renormalized inverse local time processes at the origin converge in law to some Lévy process which is not necessarily a subordinator. Indeed the corresponding strings with the origin a regular boundary converge to a string with the origin an exit boundary.

We consider non-singular conservative $\frac{d}{dm}\frac{d}{dx}$-diffusion processes and generalize the convergence theorem of Kasahara–Watanabe [1] in terms of the Poisson point fields. For this generalization we need to establish the generalized notion of the excursion measure $\boldsymbol{n}$ away from an *exit* boundary.

We have the following two well-known formulae of descriptions of usual excursion measures. One is the disintegration formula with respect to the lifetime ζ:

$$\boldsymbol{n}(\Gamma) = \int_0^\infty \boldsymbol{P}_t^{0,0}(\Gamma)\, \boldsymbol{n}(\zeta \in dt). \tag{1}$$

The other is the disintegration formula with respect to the maximum M:

$$\boldsymbol{n}(\Gamma) = \int_0^\infty \boldsymbol{R}^a(\Gamma)\, \boldsymbol{n}(M \in da). \tag{2}$$

The latter formula is often called the *Williams description*. Here $\boldsymbol{P}_t^{0,0}$ and $\boldsymbol{R}^a$ are defined through the harmonic transform of the original process. We establish these two formulae (1) and (2) for our generalized excursion measures.

Let $\boldsymbol{N}(m; dt, de)$ denote the Poisson point field with intensity mesaure $dt\,\boldsymbol{n}(de)$ and let $\widetilde{\boldsymbol{N}}(m; dt, de)$ be its compensated random field. We consider a process defined by

$$U[f](m;t) = \int_{\{\zeta<1\}} f(\zeta(e))\widetilde{\boldsymbol{N}}(m;(0,t],de) + \int_{\{\zeta\geq 1\}} f(\zeta(e))\boldsymbol{N}(m;(0,t],de). \tag{3}$$

We establish the continuity theorem with respect to the string m:

$$U[f](m_n;t) \xrightarrow{\text{law}} U[f](m;t) \tag{4}$$

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as m_n converges to m. If $f(x) \equiv x$, then the expression (3) gives the compensated inverse local time processes. Our continuity theorem 4 provides a generalization of the convergence theorem of Kasahara–Watanabe [1] in terms of the Poisson point fields.

The result of Watanabe [2] was based on Kasahara's continuity theorem of Krein's correspondence between a string and its spectral measure in the class of strings where the left boundary is regular. Kasahara and Watanabe [1] and Kotani have generalized the continuity theorem to the class of dual strings σ^* where the left boundary is entrance and of limit circle type.

Our excursion measure is defined through the spectral measure θ of a string where the origin is exit. We establish the relation between the two spectral measures θ and σ^* as follows:

$$\theta(d\xi) = \xi\sigma^*(d\xi). \tag{5}$$

This result unifies the framework of our generalized excursion measure in terms of θ with that of Kasahara–Watanabe [1] and Kotani in terms of σ^*.

References

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