Heat Kernel Estimates for Jump Processes of Mixed Types on Metric Measure Spaces

Zhen-Qing Chen (Seattle) and **Takashi Kumagai (Kyoto)**

In this talk, we investigate symmetric jump-type processes on some class of metric measure spaces with jumping intensities comparable to radially symmetric functions on the space. A typical example is the symmetric jump process on $\mathbb{R}^d$ with jumping intensity

$$\int_{\alpha_1}^{\alpha_2} \frac{c(\alpha, x, y)}{|x-y|^{d+\alpha}} \, \nu(d\alpha),$$

where ν is a probability measure on $[\alpha_1, \alpha_2] \subset (0,2)$, and $c(\alpha, x, y)$ is a jointly measurable function that is symmetric in (x, y) and is bounded between two positive constants. We establish parabolic Harnack principle and derive sharp two-sided heat kernel estimate for such jump-type processes.

Framework and Results.
Let (F, ρ, μ) be a metric measure space. Assume that F is locally compact and separable, ρ is the metric, and μ is the Radon measure. Assume further that $\overline{B(x,r)}$ is compact for all $x \in F, r > 0$.

We assume the following uniform volume doubling condition: there exists strictly increasing function $V : \mathbb{R}_+ \to \mathbb{R}_+$ such that there are $c_1, c_2 > 1$ so that $V(0) = 0,\ V(2r) \le c_1 V(r)$ for all $r > 0$ and

$$c_2^{-1} V(r) \le \mu(B(x,r)) \le c_2 V(r) \qquad \forall x \in F,\ r > 0.$$

Let

$$\mathcal{E}(f,f) = \int\int_{F \times F} (u(x) - u(y))^2 J(x,y) \mu(dx) \mu(dy),$$

where $J(x,y) \ge 0$ is symmetric measurable.

We assume

$$J(x,y) \asymp \frac{1}{V(\rho(x,y)) \phi(\rho(x,y))}, \tag{1}$$

where $f \asymp g$ means $c_1 f \le g \le c_2 f$. Here $\phi : \mathbb{R} \to \mathbb{R}_+$ is strictly increasing and there exist $0 < \beta' \le \beta < \infty$ and $0 < M < \infty$ such that

$$c_1 (\frac{R}{r})^{\beta'} \le \frac{\phi(R)}{\phi(r)} \le c_2 (\frac{R}{r})^{\beta} \qquad 0 < \forall r < R. \tag{2}$$

$$\frac{\phi(r)}{r^2} \int_0^r \frac{s}{\phi(s)} \, ds \le M \qquad \forall r > 0. \tag{3}$$

Proposition. Let $\mathcal{D}(\mathcal{E}) := \{f \in C_0(F) : \mathcal{E}(f) < \infty\}$ and $\mathcal{F} := \overline{\mathcal{D}(\mathcal{E})}^{\mathcal{E}_1}$. Then, $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(F, \mu)$.

In the following, we assume the following: there exists a metric space $X \supset F$ such that $\rho(\cdot, \cdot)$ can be extended to X with dilation for F.
I.e. for every $x, y \in F$ and every $\delta > 0$, we have $\delta^{-1}x, \delta^{-1}y \in X$ and

$$\rho(\delta^{-1}x, \delta^{-1}y) \asymp \delta^{-1}\rho(x, y). \tag{4}$$

Main Theorem. Under (1), (2), (3) and (4), there exists a continuous heat kernel $p_t(x, y)$ for $(\mathcal{E}, \mathcal{F})$ such that

$$\begin{aligned} & C^{-1}\left(\frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(\rho(x,y))\phi(\rho(x,y))}\right) \\ \leq \; & p_t(x,y) \leq C\left(\frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(\rho(x,y))\phi(\rho(x,y))}\right), \end{aligned}$$

for every $t > 0, x, y \in F$, where ϕ^{-1} is the inverse function of ϕ.

Remark. One can rewrite

$$\begin{aligned} \Phi(t, \rho(x,y)) &:= \frac{1}{V(\phi^{-1}(t))} \wedge \frac{t}{V(\rho(x,y))\phi(\rho(x,y))} \\ &= \frac{1}{V(\phi^{-1}(t))}\left\{1 \wedge \left(\frac{V(\phi^{-1}(t))}{V(\rho(x,y))}\frac{t}{\phi(\rho(x,y))}\right)\right\} \\ &= \begin{cases} \frac{1}{V(\phi^{-1}(t))} & \text{if } \phi^{-1}(t) \geq \rho(x,y) \\ \frac{t}{V(\rho(x,y))\phi(\rho(x,y))} & \text{if } \phi^{-1}(t) \leq \rho(x,y). \end{cases} \end{aligned}$$

Examples. 1) $[\alpha_1, \alpha_2] \subset (0, 2)$, μ: probability measure on $[\alpha_1, \alpha_2]$

$$\phi(t) := \int_{\alpha_1}^{\alpha_2} t^{\alpha}\, \nu(d\alpha).$$

2) $[\alpha_1, \alpha_2] \subset (0, 2)$, μ: probability measure on $[\alpha_1, \alpha_2]$

$$\phi(t) := \left(\int_{\alpha_1}^{\alpha_2} t^{-\alpha}\, \nu(d\alpha)\right)^{-1}.$$

Especially, $0 < \alpha_1 < \cdots < \alpha_n < 2$,

$$J(x,y) = \sum_{k=1}^{n} \frac{c_i(x,y)}{V(\rho(x,y))\rho(x,y)^{\alpha_i}},$$

where $c^{-1} < c_i(x,y) = c_i(y,x) < c$.