Semiclassical limit of the lowest eigenvalue of Schrödinger operators on path spaces

Shigeki Aida
Osaka University

Let (X, g) be a compact Riemannian manifold and E, V be smooth real-valued functions on X. Let μ_λ be the probability measure such that $d\mu_\lambda = Z_\lambda^{-1} e^{-\lambda E} dx$ on X, where dx is the volume element and Z_λ is the normalizing constant.

Let

$$\mathcal{E}_\lambda(f,f) = \int_X |\nabla f(x)|^2 d\mu_\lambda(x) = \int_X (-L_\lambda f)(x) \cdot f(x) d\mu_\lambda(x) \tag{1}$$

$$\mathcal{E}_{\lambda,V}(f,f) = \mathcal{E}_\lambda(f,f) + \lambda^2 \int_X V(x) f(x)^2 d\mu_\lambda(x) \tag{2}$$

$$= \int_X (-L_{\lambda,V} f)(x) \cdot f(x) d\mu_\lambda(x) \tag{3}$$

$$E_0(\lambda, V) = \inf \left\{ \mathcal{E}_{\lambda,V}(f,f) \;\middle|\; \|f\|_{L^2(\mu_\lambda)} = 1 \right\}. \tag{4}$$

We recall the asymptotic behavior of $E_0(\lambda, V)$ as $\lambda \to \infty$. Assume that
(A1) $U(x) = \frac{|\nabla E(x)|^2}{4} + V(x)$ is a nonnegative function which has finite zero point set $\{c_1, \ldots, c_n\}$.
(A2) The Hessian of U at c_i $(1 \le i \le n)$ is strictly positive.

Then we have

$$\lim_{\lambda \to \infty} \frac{E_0(\lambda, V)}{\lambda} = \min_{1 \le i \le n} \operatorname{tr} \left\{ \sqrt{\frac{(\nabla^2 U)(c_i)}{2}} - \frac{\nabla^2 E}{2}(c_i) \right\}, \tag{5}$$

where ∇^2 denotes the second covariant derivative which is defined by the Levi-Civita connection. We extend this result to path space X with a Dirichlet form and a probability measure μ_λ which is written formally as $d\mu_\lambda(\gamma) = Z_\lambda^{-1} \exp(-\lambda E(\gamma))\, d\gamma$, where γ denotes a path and $E(\gamma)$ is the energy of the path. $d\gamma$ is a formal Riemannian volume. In this talk, we consider the following three cases.

(I) X is an abstract Wiener space (B, H, μ). The Dirichlet form is given by

$$\mathcal{E}_{\lambda,A}(f,f) = \int_B |A(w) Df(w)|_H^2 d\mu_\lambda(w). \tag{6}$$

Here D denotes the usual H-derivative, $A(w) \in L(H,H)$ and $\mu_\lambda(\cdot) = \mu(\sqrt{\lambda}\cdot)$. We consider the Schrödinger operator $-L_{\lambda,A,V}$ corresponding to a semi-bounded form $\mathcal{E}_{\lambda,A,V}(f,f) = \mathcal{E}_{\lambda,A}(f,f) + \lambda^2 \int_B V(w) f(w)^2 d\mu_\lambda(w)$. In this case, $E(w) = \frac{1}{2}\|w\|_H^2$. If $A(w) = I_H$, then the limit $\lim_{\lambda\to\infty} \frac{E_0(\lambda,V)}{\lambda}$ was studied in [1].

(II) $X = P_x(M) = C([0,1] \to M \mid \gamma(0) = x)$. Here M is a compact Riemannian manifold. The measure μ_λ is the Brownian motion measure which is given by the heat semigroup $e^{\frac{t}{2\lambda}\Delta}$. The Dirichlet form is given by the H-derivative:

$$(\nabla F)(\gamma) = \sum_{i=1}^{n} \tau(\gamma)_{t_i}^{-1} (\nabla f)_{\gamma(t_i)}(\gamma(t_1), \ldots, \gamma(t_n)) t \wedge t_i, \tag{7}$$

where $F(\gamma) = f(\gamma(t_1), \ldots, \gamma(t_n))$ and $\tau(\gamma)_t : T_x(M) \to T_{\gamma(t)}M$ denotes the stochastic parallel translation. In this case, $E(\gamma) = \frac{1}{2}\int_0^1 |\dot\gamma(t)|^2 dt$.
(III) $X = P_{e,a}(G) = C([0,1] \to G \mid \gamma(0) = e, \gamma(1) = a \in G)$. Here G is a compact Lie group, e is the unit element. In this case, μ_λ is the pinned Brownian motion measure $\nu_{\lambda,e,a}$ which is defined by $e^{\frac{t}{2\lambda}\Delta}$. The Dirichlet form is given by the probability measure $\nu_{\lambda,e,a}$ and the H-derivative:

$$(\nabla f(\gamma), h) = \lim_{\varepsilon \to 0} \frac{f(e^{\varepsilon h(\cdot)}\gamma(\cdot)) - f(\gamma)}{\varepsilon}, \tag{8}$$

where $h \in H^1([0,1] \to \mathfrak{g} \mid h(0) = h(1) = 0)$ and $\mathfrak{g}$ is the Lie algebra of G. In this case again, $E(\gamma) = \frac{1}{2}\int_0^1 |\dot\gamma(t)|^2 dt$.

In these cases, we can determine the limit $\lim_{\lambda\to\infty} \frac{E_0(\lambda,V)}{\lambda}$ under the similar assumptions in (A1) and (A2). In the proof of the lower bound estimate, we combine the following two results:
(1) Rough lower bound estimate on $E_0(\lambda, V)$ which is given by a log-Sobolev inequality on X
(2) "Approximation" of the Schrödinger operator near zero points of the potential function U by a Schrödinger operator with a quadratic potential function on an abstract Wiener space in the case where $A(w) = I_H + T$ and T is a trace class operator which is independent of w.

In the case of (II)($X = P_x(M)$), as to (2), we use an infinite dimensional version of the following simple result and a pointwise estimate on $\frac{L_\lambda \Omega_{\lambda,V}(x) - L_{\lambda,A}\Omega_{\lambda,V}(x)}{\Omega_{\lambda,V}(x)}$.

Proposition 1 *We consider the forms in* (1) *and* (2) *and the lowest eigenvalue* $E_0(\lambda, V)$ *on the compact Riemannian manifold* (X, g). *Let* $A(x) \in L(T_xX, T_xX)$ *and assume that* $x \to A(x)$ *is smooth and set*

$$\mathcal{E}_{\lambda,A}(f,f) = \int_X |A(x)\nabla f(x)|^2 d\mu_\lambda(x) = \int_X (-L_{\lambda,A}f)(x) \cdot f(x) d\mu_\lambda(x) \tag{9}$$

$$\mathcal{E}_{\lambda,A,V}(f,f) = \mathcal{E}_{\lambda,A}(f,f) + \lambda^2 \int_X V(x) f(x)^2 d\mu_\lambda(x). \tag{10}$$

Let $\Omega_{\lambda,V}$ *be the posivive normalized eigenfunction of* $-L_{\lambda,V}$ *corresponding to* $E_0(\lambda,V)$. *Then for any* $f \in C^\infty(X)$, *we have*

$$\mathcal{E}_{\lambda,A}(f\Omega_{\lambda,V}, f\Omega_{\lambda,V}) = \int_X |A(x)\nabla f(x)|^2 \Omega_{\lambda,V}(x)^2 d\mu_\lambda(x) + E_0(\lambda,V)\|f\Omega_{\lambda,V}\|^2_{L^2(\mu_\lambda)} + \int_X \frac{L_\lambda\Omega_{\lambda,V}(x) - L_{\lambda,A}\Omega_{\lambda,V}(x)}{\Omega_{\lambda,V}(x)} f(x)^2 \Omega_{\lambda,V}(x)^2 d\mu_\lambda(x). \tag{11}$$

References

[1] S. Aida, Semiclassical limit of the lowest eigenvalue of a Schrödinger operator on a Wiener space, J. Funct. Anal.**203**, (2003), 401–424.