Gaugeability with Applications to Symmetric $\alpha$-Stable Processes
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The purpose is to give two applications of gaugeability to symmetric $\alpha$-stable processes.

$M = (\Omega, P_x, X_t)$: a symmetric $\alpha$-stable process on $\mathbb{R}^d$ ($0 < \alpha < 2$).

$(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$: the Dirichlet form of $M^\alpha$.

$\text{Cap}(A)$: the capacity of a set $A$ defined by $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$.

For an open set $D \subset \mathbb{R}^d$, $M^D = (P_x, X^D_t)$: the absorbing process on $D$.

Assume that $M^D$ is transient, that is, $\text{Cap}(\mathbb{R}^d \setminus D) > 0$ if $\alpha \geq d$.

$G_D(x, y)$: the Green function of $M^D$.

**Definition 1.**
A positive Radon measure $\mu$ on $\mathbb{R}^d$ is said to be in the class $S^D_\infty$, if for any $\epsilon > 0$ there exists a compact set $K \subset D$ and $\delta > 0$ such that

$$\sup_{(x, z) \in D \times D \setminus K^c} \int_{K^c} \frac{G_D(x, y)G_D(y, z)}{G_D(x, z)} \mu(dy) \leq \epsilon$$

and for all measurable sets $B \subset K$ with $\mu(B) < \delta$,

$$\sup_{(x, z) \in D \times D \setminus B} \int_B \frac{G_D(x, y)G_D(y, z)}{G_D(x, z)} \mu(dy) \leq \epsilon$$

$\mu \in S^D_\infty \iff$ PCAF $A^\mu_t$. (Revus Correspondence)

For a measure $\mu$ in $S^D_\infty$, define

$$\lambda(\mu; D) = \inf \left\{ \mathcal{E}^{(\alpha)}(u, u) : u \in C^\infty_0(D), \int_D u^2(x)\mu(dx) = 1 \right\}.$$}

$p_t^{\mu,D}(x, y)$: the integral kernel of the Feynman-Kac semigroup,

$$p_t^{\mu,D}f(x) := \mathbb{E}_x[\exp(A^\mu_t)f(X_t); t < \tau_D] = \int_D p_t^{\mu,D}(x, y)f(y)dy$$

$G^{\mu, D}(x, y)$: the Green function, $G^{\mu,D}(x, y) = \int_0^\infty p_t^{\mu,D}(x, y)dt$.  

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Theorem 1. (Z. Zhao, T, Z. Q. Chen)
Let $\mu \in S^D_\infty$. Then the following conditions are equivalent:

(i) (gaugeability) $\sup_{x \in D} \mathbb{E}_x [e^{A^{\mu}_t D}] < \infty$

$\iff \exists x_0 \in D \text{ s.t. } \mathbb{E}_{x_0} [e^{A^{\mu}_t D}] < \infty$;

(ii) (subcriticality) $G^{\mu,D}(x, y) < \infty$ for $x, y \in D, x \neq y$;

(iii) $\lambda(\mu; D) > 1$.

Applications

(i) The first is relevant with the ultracontractivity of Schrödinger semigroups $p^\mu_t$:

$$p^\mu_t f(x) = \mathbb{E}_x [\exp(A^\mu_t) f(X_t)].$$

$\|p^\mu_t\|_{1,\infty}$: the operator norm of $p^\mu_t$ from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$.

Theorem 2.
Let $\mu \in S_\infty$ with $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^{\alpha-d} d\mu(x)d\mu(y) < \infty$. Then

$$\lambda(\mu; \mathbb{R}^d) > 1 \iff \|p^\mu_t\|_{1,\infty} \leq \frac{c}{t^{d/\alpha}}, \ t > 0.$$ 

(ii) $\mathbb{B}^\alpha = (\bar{X}_t, \bar{P}_x)$: the branching $\alpha$-symmetric stable process with the branching rate $k$, a smooth measure of $\mathbb{M}^{(\alpha)}$,

$$\bar{P}_x[T > t|\sigma(X)] = \exp(-A^k_t),$$

where $T$ is the first splitting time.

$\{p_n(x)\}_{n \geq 2}$: the branching mechanism

$Q(x) = \sum_{n \geq 2} np_n(x), \mu(dx) = (Q(x) - 1)k(dx)$.
Assume that $\sup_{x \in \mathbb{R}^d} Q(x) < \infty$.

**Theorem 3.**  
Let $K$ be a closed set with $\text{Cap}(K) > 0$. If $\mu \in \mathcal{S}_{\infty}^{\mathbb{R}^d \setminus K}$, then

$$\lambda(\mu; \mathbb{R}^d \setminus K) > 1 \iff \mathbb{E}_x[N_K] < \infty.$$  

Here $N_K$ is the number of branches of $B^\alpha$ ever hitting $K$.

**References**


