Gaugeability with Applications to Symmetric α-Stable Processes

Takeda Masayoshi, Tohoku University, Mathematical Institute

The purpose is to give two applications of gaugeability to symmetric α-stable processes.
$\mathbb{M} = (\Omega, \mathbb{P}_x, X_t)$: a symmetric α-stable process on $\mathbb{R}^d$ $(0 < \alpha < 2)$.
$(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$: the Dirichlet form of $\mathbb{M}^\alpha$.
Cap(A): the capacity of a set A defined by $(\mathcal{E}^{(\alpha)}, \mathcal{D}(\mathcal{E}^{(\alpha)}))$.
For an open set $D \subset \mathbb{R}^d$, $\mathbb{M}^D = (\mathbb{P}_x, X_t^D)$: the absorbing process on D

Assume that $\mathbb{M}^D$ is transient, that is, $\mathrm{Cap}(\mathbb{R}^d \setminus D) > 0$ if $\alpha \geq d$

$G_D(x, y)$: the Green function of $\mathbb{M}^D$.

Definition 1.
A positive Radon measure μ on $\mathbb{R}^d$ is said to be in the class $\mathcal{S}_\infty^D$, if for any $\epsilon > 0$ there exists a compact set $K \subset D$ and $\delta > 0$ such that

$$\sup_{(x,z)\in D\times D\setminus d} \int_{K^c} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \mu(dy) \leq \epsilon$$

and for all measurable sets $B \subset K$ with $\mu(B) < \delta$,

$$\sup_{(x,z)\in D\times D\setminus d} \int_{B} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \mu(dy) \leq \epsilon$$

$$\mu \in \mathcal{S}_\infty^D \Longleftrightarrow \text{PCAF } \mathrm{A}_\mathrm{t}^\mu. \quad (\textbf{Revus Correspondence})$$

For a measure μ in $\mathcal{S}_\infty^D$, define

$$\lambda(\mu; D) = \inf \left\{ \mathcal{E}^{(\alpha)}(u,u) : \ u \in C_0^\infty(D), \int_D u^2(x)\mu(dx) = 1 \right\}.$$

$p_t^{\mu,D}(x,y)$: the integral kernel of the Feynman-Kac semigroup,

$$p_t^{\mu,D} f(x) := \mathbb{E}_x[\exp(A_t^\mu) f(X_t); t < \tau_D] = \int_D p_t^{\mu,D}(x,y) f(y) dy$$

$G^{\mu,D}(x,y)$: the Green function, $G^{\mu,D}(x,y) = \int_0^\infty p_t^{\mu,D}(x,y)dt$.

Theorem 1.(Z.Zhao, T, Z.Q.Chen)
Let $\mu \in \mathcal{S}_\infty^D$. Then the following conditions are equivalent:

(i) (**gaugeability**) $\sup_{x \in D} \mathbb{E}_x[e^{A^\mu_{\tau_D}}] < \infty$
$\left(\Longleftrightarrow \exists x_0 \in D \ \text{ s.t. } \mathbb{E}_{x_0}[e^{A^\mu_{\tau_D}}] < \infty \right)$;

(ii) (**subcriticality**) $G^{\mu,D}(x,y) < \infty$ for $x, y \in D$, $x \neq y$;

(iii) $\lambda(\mu; D) > 1$.

Applications

(i) The first is relevant with the **ultracontractivity** of Schrödinger semigroups p_t^μ:

$$p_t^\mu f(x) = \mathbb{E}_x[\exp(A_t^\mu) f(X_t)].$$

$\|p_t^\mu\|_{1,\infty}$:the operator norm of p_t^μ from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$.

Theorem 2.
Let $\mu \in \mathcal{S}_\infty$ with $\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x-y|^{\alpha-d} d\mu(x) d\mu(y) < \infty$. Then

$$\lambda(\mu; \mathbb{R}^d) > 1 \Longleftrightarrow \|p_t^\mu\|_{1,\infty} \leq \frac{c}{t^{d/\alpha}}, \quad t > 0.$$

(ii) $\mathbb{B}^\alpha = (\bar{X}_t, \bar{\mathbb{P}}_x)$: the branching α-symmetric stable process with the branching rate k, a smooth measure of $\mathbb{M}^{(\alpha)}$,

$$\bar{\mathbb{P}}_x[T > t | \sigma(X)] = \exp(-A_t^k),$$

where T is the first splitting time.
$\{p_n(x)\}_{n \geq 2}$: the branching mechanism
$Q(x) = \sum_{n \geq 2} n p_n(x)$, $\mu(dx) = (Q(x) - 1) k(dx)$.

Assume that $\sup_{x\in\mathbb{R}^d} Q(x) < \infty$.

Theorem 3.
Let K be a closed set with $\mathrm{Cap}(K) > 0$. If $\mu \in \mathcal{S}_\infty^{\mathbb{R}^d\setminus K}$, then

$$\lambda(\mu; \mathbb{R}^d \setminus K) > 1 \Longleftrightarrow \bar{\mathbb{E}}_x[N_K] < \infty.$$

Here N_K is the number of branches of $\mathbb{B}^\alpha$ ever hitting K.

References

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